

# TEACHING AND LEARNING ALGEBRA : APPROACHING COMPLEXITY THROUGH COMPLEMENTARY PERSPECTIVES

Michèle Artigue  
University of Paris 7, DIDIREM  
<artigue@math.jussieu.fr >  
Brigitte Grugeon  
IUFM d'Amiens & DIDIREM  
<brigitte.grugeon@amiens.iufm.fr >

Teresa Assude  
IUFM de Versailles & DIDIREM  
< assude@math.jussieu.fr >  
Agnès Lenfant  
IUFM de Reims & DIDIREM  
< agnes.lenfant@infonie.fr >

Abstract : In this submission, after presenting three different research projects on algebra, we have carried out or are developing in our research team DIDIREM, at the Université Paris 7, we try to summarise what can be offered by this kind of didactic research to the ICMI Study and stress the necessity of connecting different approaches if we want to better understand the complexity of learning and teaching processes in that area and improve them in a rational way.

## I. Introduction

Algebra is a crucial domain as regards the relationships students develop with mathematics. For a lot of these, and for most adults in society, algebra is the domain where, abruptly, mathematics became a non understandable world. Faced with such evident teaching and learning difficulties, didactic research has been very active during the last twenty years. It firstly tried to better understand learning processes in algebra and explain the breach mentioned above. These attempts were successful in identifying some decisive factors, such as those linked to the discontinuities existing between arithmetic and algebraic thinking modes and the specificity of algebraic semiotic practices. Didactic research also developed insightful analysis of usual teaching practices in that area, in various countries, and helped us explain their observed inefficiency. More recently, research tried to explore the potential offered by computer technologies in order to overcome the identified learning difficulties and to develop more effective teaching strategies. Books such as (Bednarz, Kieran, Lee, 1996) fairly well illustrate the richness of the research work undertaken up to now and the coherence of its results, in spite of the evident diversity of the theoretical approaches and contexts.

Our research team, DIDIREM, has been involved in didactic research in algebra for nearly ten years and we would like to rely on different pieces of research we have carried out or are carrying out, in order to contribute to the ICMI Study on Algebra. In the first part of this collective contribution, we present the way research developed, the corresponding “problématiques” and the associated theoretical frames. Then, we briefly describe three different research projects. Finally, in the last part, we discuss what can be offered by such a research work to the reflection on the future of learning and teaching algebra.

## II. Development of research and of its theoretical frames

In our team, didactic research on algebra began to develop in three independent directions : the teaching and learning of linear algebra at university level, the analysis of

institutional transitions in algebra and the analysis of potential offered by CAS for teaching and learning elementary algebra. This contribution mainly focuses on the work undertaken in the second direction and its subsequent developments<sup>1</sup>. Through these projects, we became more and more sensitive to the complexity of this research domain and to the necessity of approaching it through different complementary but coherent approaches.

The complexity of this research domain is linked to the fact that understanding learning and teaching processes in algebra, even if one restricts to middle and high school algebra as is the case in this contribution, doesn't only require the understanding of students' learning processes seen as pure cognitive processes. This is more: this is understanding how the scientific and technological evolution influences algebraic knowledge and practices in the large today, how it changes the cultural, social and professional needs as regards algebra ; this is understanding the functioning of different complex institutional systems where algebra is taught and learnt, taking into account their respective constraints, traditions and cultures ; this is understanding teachers' culture in algebra, expectations and practices and their potential effects on students' learning ; this is understanding the way curriculum and syllabus develop, the role played by textbooks and other traditional didactic resources but also what could be offered by resources which play an increasing role such as software, CDROMs, websites...

In our research team, we try to explore and co-ordinate complementary approaches to this complexity and, in this contribution, we rely on three different projects: the initial project mentioned above about institutional transitions and its further extensions, a project on teacher professional development in algebra, and a project developing an historical perspective in order to analyse the curricular evolution and thus better understand present curricula as the result of a specific history, within a specific culture. In the following, we briefly introduce our "problématiques" and theoretical frames through the narration of the birth of the first project. At a macro-didactic level, we use the anthropological approach to didactic phenomena developed by Y. Chevallard (1992) as a global theoretical perspective to which each of our more local perspectives can be related in a coherent way.

The transition project emerged from an educational problem: in France, there exist bridges between vocational and general higher education and best students coming from vocational high schools can enter a specific one year course designed for helping the transition process. In spite of this, transition remains specially difficult and algebra plays a crucial role in its failure. Generally, such transition problems are considered as resulting from the low mathematical level of students coming from vocational high schools. B. Grugeon, who taught transition classes, hypothesised that this general interpretation was too convenient and certainly a simplistic one. She decided to approach this transition issue within the anthropological approach mentioned above. According to this approach, mathematical knowledge cannot be considered as something absolute. It strongly depends on the institutions where it has to live, to be learnt, to be taught. Mathematical objects do not exist per se but emerge from practices which are different from one institution to another one. Y. Chevallard analyse these in terms of "praxeologies", that is to say in terms of tasks, techniques used to solve these tasks, "technologies" which denote the discourse developed in order to explain and justify particular techniques, and last, "theories" that he defines as technologies for the technologies, which organise the local technological discourse in coherent structures. Within this approach, the familiar sentences : "(s)he knows that" or "(s)he doesn't know that", do not make sense, taken as such. Every institution which has to deal with some mathematical object develops an institutional relationship with this object. This relationship defines the norms and values of knowledge as regards this object, for this particular institution. Institutional relationships with

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<sup>1</sup> The results obtained in the first direction are presented in the book edited by J.L. Dorier and published by Kluwer in 2000 : "On the teaching of linear algebra", and an independent contribution which integrates the main results obtained in the third direction is proposed by J.B. Lagrange.

one mathematical object or domain vary across institutions. Students, exposed to teaching or other socio-cultural experiences develop personal relationships to mathematical objects, which are shaped by the different institutional influences they are submitted to. This is only if their personal relationship with an object is close enough from the institutional relationship at stake that they are considered by one particular institution as “knowing this object”. This theoretical frame led B. Grugeon to the following conjecture: vocational and general high schools have quite different institutional relationships with elementary algebra and the poor sensitivity of the educational system to these discrepancies could explain part of the students’ difficulties in the transition process. She also supposed that being aware of such a fact, could allow the development of efficient didactic tools. This was the starting point for the research we describe in the third part of this text (Grugeon, 1995).

### III. Institutional transitions

In order to test the initial hypothesis we have just mentioned, it was necessary to define a kind of external reference with respect to elementary algebra, independent from institutional values. This was achieved through the elaboration of a multidimensional structure of analysis for elementary algebraic knowledge, based on existing research in that area.

The field of algebraic knowledge can be structured around two main non-independent dimensions: the tool and object dimensions (Douady, 1984). More precisely, in its tool dimension, algebraic competence can be described through the ability to mobilise algebraic tools in order to solve different kind of problems internal or external to the mathematics field (at the elementary level considered here, problems of generalisation and proof, traditional arithmetical problems, problems where algebra appears as a modelling tool, algebraic and functional problems). In its object dimension, that is to say considering algebra as a structured set of objects with specific properties, semiotic representations, treatment modes, algebraic competence can be described through the ability to cope with algebraic objects, by taking account both their semantics and their syntax. At the level of schooling considered here, specific attention is put on the arithmetical / algebraic cut, on the ability to interpret algebraic expressions both at operational and structural level, and on the ability to flexibly adapt algebraic interpretations in order to pilot algebraic work.

#### *The multidimensional grid of analysis*

This analysis resulted in a multidimensional grid structured around six components, each of these being specified by a set of criteria with corresponding potential values, which was then used in order to analyse personal and institutional relationships to algebra. The six component are the following:

The first component has a specific role. From an institutional point of view, she allows to compare the types of problems favoured by a given institution ; from a personal point of view, it allows to evaluate the algebraic competence with respect to given institutional norms. Nine type of algebraic treatment are a priori identified, both according to the tool and object dimensions. They are only partially ordered.

The five remaining components aim at identifying and describing important characteristics, local coherences both in official syllabus and students’ algebraic functioning. They are the following : (1) relationship between arithmetic and algebra, approached through the following criteria : resolution process, status of equality sign, status of letters, objects and status of these; (2) processing of algebraic expressions ; (3) connections between the algebraic symbolic register and other semiotic registers, where, according to Duval’s work distinctions are made between the formation of expressions and their processing, and between the nature of conversions between registers ; (4) functionalities of algebra, and (5) algebraic rationality.

The analysis of personal relationships was achieved through the elaboration of a set of 19 diagnostic tasks, we cannot describe here. The set was conceived in order to allow the

identification of local coherence in students' algebraic behaviour. Thus, the value for each criteria in the grid could be assessed at least five times. The first experimentation showed that the initial values introduced for the different criteria were insufficient for covering the diversity of students' behaviour and the diagnostic potential of each task of the set. Thus local values were added and the second experimentation, one year later, showed that the new version of the grid was adequate.

### *Essential results*

Institutional relationships were classically explored through the analysis of syllabus, textbooks, national or regional assessment tasks, but also the careful analysis of students' notebooks corresponding to their last year in vocational high school. Institutional analysis evidences differences in algebraic culture between the two institutions at stake, all the more pernicious as a quick look at the different syllabuses let the impression that the vocational syllabus is built from the general high school syllabus by cutting and pasting, then adding some specific professional uses related to finance and accountancy. They thus look very close. A more detailed analysis of the syllabuses, using the multidimensional grid, makes visible a conjunction of small differences which, taken together, characterise two very different cultures. The notebook analysis confirms this fact, while showing that the differences between syllabuses are reinforced by differences in the more global relationship the two institutions develop with respect to mathematics.

Analysis of personal relationships through the set of diagnostic tasks allowed us to build a cognitive description for each students. The micro-description made from the 19 n-uplets associated with their answers to the diagnostic could not be used as such. So, a macro-description, giving a synthetic vision, was built by identifying categories of answers, component by component. This resulted, for each student, in a quantitative description of her or his algebraic competences (first component), a qualitative identification of coherences in the algebraic functioning, and a qualitative description of his or her flexibility in connecting the symbolic system of algebra with other semiotic registers. This was expressed in terms of students' profiles. In the research process, these students' profiles were used in order to partially individualise teaching, taking into account the specific relationship each student had with algebra, a didactic strategy which resulted to be very positive for the great majority of students.

Thanks to this research work, we also became progressively aware of the diversity of possible germs for an entrance in algebraic thinking. High school teaching favours some of these at the expense of many possible other ones, and especially tends to underestimate the algebraic work with formulas which is so important in the vocational culture. Research has shown that, through adequate strategies, this entrance can be a very fruitful one if formulas are not only objects given to the student and if the technical work that formula support is not too reduced in complexity.

### *The subsequent developments of this research work*

This research was first extended to the study of other institutional transitions, and especially the transition between junior and senior at school (grades 9 / 10). Both the initial grid and the set of diagnostic task was adapted for this purpose, but this adaptation did not rise important difficulties and it was soon achieved.

The next ambition of the group was to make the efficient tools of analysis elaborated in the framework of this research useful beyond the sole community of researchers. A first simplified version of the grid was tested with a small sample of teachers with different experience (Lenfant, 1997). The results were very disappointing. Coding students' response and then interpreting codes in terms of profiles for devising appropriate didactic actions was too complex and time consuming. For overcoming this evident obstacle, a new project was developed in collaboration with researchers in artificial intelligence. It aimed at creating a computer version of the diagnostic and at computerising coding and analysis up to the building

of students' profiles. This was called the PEPITE project which resulted in the S. Jean's doctoral thesis (Jean, 2000). Transposing the paper version of the diagnostic into a computer version was not easy at all, as adequate interface and transpositions of the tasks had to be built. Successive versions were developed and tested with students. There is no doubt that working in a computer environment modifies the relationship students have with the diagnostic set, their means of work and expression, their strategies. Nevertheless, successive adaptations led to the point where these differences did not reduce the diversity of possible behaviours and did not prevent from identifying characteristics of algebraic functioning which could be also observed in standard environments. The computer version of the diagnostic (PépiTest), was then coupled with two analysis modules : PépiDiag and PépiProfil. Their elaboration was also the source of difficult problems as one can easily imagine. Successive adaptations were made, taking into account the results of comparisons between computer and hand-made codes on the same students' productions. The product is now operational and research, still in collaboration with researchers in artificial intelligence, aims at developing remedial tasks which could be proposed to students, in the same environment, after identification of their cognitive profile.

#### IV. The first steps of teachers' professional development in algebra

We cannot expect substantial improvements in the teaching and learning of algebra if we do not take into consideration the teacher dimension. As shown above, this issue became evident with B. Grugeon's attempts to make the tools of analysis she had elaborated in her doctoral thesis widely available and useful beyond the sole community of researchers. Two research projects were thus developed: on the one hand, the partial computerisation of the diagnostic and profile building described above, and, on the other hand, a research project aiming at improving our knowledge of teachers' professional knowledge in algebra and the ways it develops. This is the theme of the on-going doctoral thesis by A. Lenfant we briefly present now.

This research focuses on the professional development of pre-service teachers (in the following: PLC<sup>2</sup>) who, in France, have just passed the national competition called CAPES in order to become secondary mathematics teachers and are given one year of professional training in an IUFM (Institut Universitaire de Formation des Maîtres). During this year of professional training, they also have one class in full responsibility, 6 hours per week. So, they are moving from a student position in the educational system to a teacher position. We use algebra in order to analyse this transition process and the ways it could be more efficiently assisted through the training offered at the IUFM. Algebra is a privileged domain for such an analysis for several reasons: this domain presents evident learning difficulties which are not so easy to understand when algebraic modes of thinking and algebraic techniques have become so familiar that they are quite "naturalised", as is the case for the PLCs. Moreover, all PLCs are concerned with these difficulties as they have to teach algebra or pre-algebra in their full responsibility class.

In this research, we use the anthropological approach (Chevallard, 1997, 1999) in order to analyse the "mathematical and didactic organisations" developed by teachers with respect to algebra, the associated "mathematics and didactic praxeologies" and, more globally, the mathematical professional work of the teacher via its different professional "gestures", in and out the classroom. Professional development in algebra is seen as a complex genesis, relying on a mixture of competencies specific to this domain and of more transversal competencies which strongly intertwine in practices. We thus consider that professional competence has to be analysed in multidimensional terms. It can be modelled as a multivariate function which shapes the decisions the teacher takes in her (his) different professional gestures, the way (s)he faces unforeseen situations, the discourse and analysis (s)he develops at a more reflective level. In

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<sup>2</sup> PLC is an acronym for high school teacher (professeur de lycée et collège, in French).

order to approach it, and inspired by B. Grugeon's research work, we have developed a multidimensional grid for professional competence in elementary algebra (MGPCA in the following) which focuses on the specific competencies and tries to describe potential underlying knowledge. We hypothesise that such knowledge influences the different professional gestures in a non-uniform way and that it cannot necessarily be made explicit. One ambition of the research is to explore the complexity of relationships between knowledge and competencies, to try to find the real influence professional algebraic knowledge plays with respect to other determinants of the PLCs' behaviour, and to evidence some regularities which could help us better understand teachers' behaviour and improve training strategies.

### *The MGPCA Grid*

The grid is structured around three non-independent dimensions: the epistemological, the cognitive and the didactic ones. In the following, we synthesise the contents of knowledge which structure the grid according to each dimension.

#### *The epistemological dimension :*

Epistemological knowledge in the grid, is structured around, on the one hand, some important features of the historical development of algebra, on the other hand, the distinction between the "tool and object" dimensions of algebra, introduced in part III. As regards the first point, one essential epistemological characteristic is the complexity of the algebraic symbolic system and the difficulties of its historical development. Such knowledge can help to understand the difficulties met by present students. Another important point which arise from historical development is the extension and diversity of the algebraic domain. Such knowledge can support the change of epistemological views about algebra, allow to better understand the rationale for the progression in algebraic knowledge organised by the curriculum and, eventually discuss its pertinence. We conjecture that, at the beginning of the academic year, the PLCs are unaware of this complexity and tend to reduce algebra to the algebraic structures and theories they have been taught at university.

As regards the second point, the distinction between the tool and object facets of algebra, we conjecture that the PLCs, at the beginning of the academic year, mainly see algebra as a tool for solving problems which can be modelled in terms of equations and that, as teachers, they tend to over-emphasise the work on algebraic techniques.

#### *The cognitive dimension :*

This component deals with potential professional knowledge about learning processes in algebra. We have organised this part of the grid around four main points linked to resistant learning difficulties evidenced by didactic research: the relationships between arithmetic and algebra, the symbolic system of algebra, the relationships between different semiotic representations used in algebra, and the relationship to algebraic rationality. For this part of the grid, we especially rely on the categories introduced by B. Grugeon, which have been described in part III.

As regards this cognitive dimension, we conjecture that, at the beginning of the academic year, the PLCs are not aware of the diversity and resistance of learning difficulties in algebra, but that, through their practice and discussions with pairs, they soon become sensitive to most of these, even if they are not able to interpret them in coherent and analytic ways as research allows to do, and thus react efficiently. We also conjecture that, due to the general educational French culture which over-emphasises the links between rationality and geometry at the expense of any other one, integrating the role algebra has to play in the development of students' mathematics rationality results specially difficult.

#### *The didactic dimension :*

Knowledge relevant to these two first dimensions certainly influences the didactic and mathematical organisations, the teachers develop. But these are also shaped by what we will call

here more specific didactic knowledge: knowledge of the curriculum, of the specific goals of algebraic teaching at a given grade, of possible progressions and activities for the teaching of algebra compatible with these and of well adapted assessment tasks, knowledge of educational resources: textbooks but also publications from the IREM (Instituts de Recherche sur l'Enseignement des Mathématiques), websites (especially as regards the use of computer tools such as spreadsheets for the teaching of algebra), etc.

The MGPCA grid is then used in order to analyse the PLCs' initial relationship with algebra, and its evolution, through different questionnaires, and through the one year long following up of a selected group of PLCs. For these, a lot of data are collected: personal note boards, representative students' copy boards, assessment tasks, videos of classroom sessions and regular interviews, and the interpretation relies on the triangulation of the different analysis carried out on this material.

### *Some results*

We have considered professional development as a multidimensional and complex process, involving epistemological, cognitive and didactic changes. We are perfectly aware that what can be reached through this first year of professional training is necessarily very limited. These evident limitations make all the more important to detect possible germs for priming professional development. The results we have obtained up to now are certainly very partial but they clearly tend to show that some interesting and subtle evolution take place. All of these don't directly affect the design and management of classroom situations. At a first level, they seem more able to express in a priori and a posteriori analysis of classroom sessions, and more in a collective way than in an individual way. Results also show important differences in the accessibility of the respective parts of what can be considered today as professional expertise in algebra.

For instance, as regards the epistemological dimension, data analysis shows that the training at the IUFM seems to easily destabilise the vision of algebra as a domain reduced to the field of algebraic structures and theories. For most students, borders of algebra tend to become questionable and this helps them as expected. Nevertheless, their vision of the different tool facets of algebra remains limited and they go on over-emphasising the object dimension of algebra. Integrating a functionality of proof seems specially difficult.

As regards the cognitive dimension, training seems make them aware of the differences between the arithmetic and the algebraic processes for solving numerical problems, and through their practice, they become sensitive to students' difficulties with the symbolic system of algebra and with the conversions between semiotic registers. But, for the majority, the short training they receive does not allow internalise operational means for analysing these difficulties. As a consequence, their remedial strategies remain quite limited. Moreover, understanding that algebraic expressions have not only a syntax and an external semantics but also an internal semantics, understanding the role this semantics plays in the piloting and control of algebraic computations seems specially difficult.

As regards the didactic dimension, textbooks, and the local advisor each PLC has in the high school where (s)he teaches in full responsibility, have a predominant influence but the PLCs do not only use the textbook selected by the high school for their class. Visibly, the work they did the year before, when preparing the CAPES, which obliged them to search and select exercises on specific mathematical themes by looking at different sources has created some "habitus". Generally, the PLCs don't meet difficulties when deciding the different points they want to address, but they have more difficulties at identifying the real aims of the teaching of pre-algebra or algebra, at the grade level corresponding to their class. They also have resistant difficulties at taking into account the initial state of knowledge of their students when they come to teach algebra. Their teaching of algebra is generally not reduced to its technical component but, even at the end of the academic year, the elaboration of technological discourse seems to remain under their sole responsibility. Globally, there is a sensible evolution during the

academic year: the lesson conception evolves, the relationship to assessment evolve, but, beyond the general tendencies we have just pointed out, each evolution seems to have its idiosyncrasy.

## V. Curricular evolution through a case study : the case of inequalities

Understanding learning and teaching processes, reflecting on possible ways for improvement, also requires a careful analysis of the curricular evolution, of the cultural tradition it reflects through its permanencies and changes. Current curriculum is the product of an history which shapes today views, values and practices, and which conditions the viability of intended change, up to a certain point. In our team, historical analysis of curricular evolution was first developed about calculus and resulted to be specially productive in order to better understand the present state of crisis of secondary teaching in that area (Artigue, 1996). Research in algebra, within this “problématique”, is more recent and benefits the conceptual tools provided by the development of research on ecology of knowledge (Artaud, 1997).

The starting point was the following: in France, the word "algebra" disappeared from the official syllabus for grades 7 to 9, the grades where traditionally algebra is introduced, more than ten years ago. Algebraic contents and practices are still present at these grades, but they are no longer grouped in a specific section of the syllabus which, from grade 6, is organised in three sections : “numerical works”, “geometrical works”, “data organisation and processing, and functions”. Such a curricular change is far from being neutral as was pointed out by Y. Chevallard, already in 1985 (Chevallard, 1985). How does it affect the teaching and the learning of algebra in the long range<sup>3</sup>? In the on-going research we present here, this issue is approached through a case study: the case of inequalities at junior high school level.

The curricular evolution at junior high school level in France, as regards this particular object, can be divided into three main stages: the first one covers the first half of the 20<sup>th</sup> century from the 1902 reform to the "new math" reform in 1970 ; the second one runs from the "new math" reform to the 1977 counter-reform ; the third one goes until now and its main characteristics were fixed by the 1985 reform. In the first stage, the context of equations is predominant ; in the second stage, this predominance is taken by the “functional and structural context” ; in the third stage, the situation is not so clear and we are faced with a mixture of empirical, structural and equation contexts. Let us first specify these three contexts.

### *The initial “equation stage”*

During this first stage, inequalities are present in the algebraic part of the syllabus for grades 8 and 9, which is divided into three parts: "arithmetic", "algebra" and "geometry". For instance, in the 1902-1905 syllabus, inequalities come after operations on positive numbers, monomials, polynomials, and the solving of equations (first grade equations with one unknown, systems of two equations with two unknowns, systems of equations with more than two unknowns). The ecological “habitat” (Artaud, 1997) for inequalities is thus that of equations. Inequalities are nearly considered as equations : definitions, algebraic techniques for their solving are derived from the corresponding ones for equations, the essential objects in the algebra part of the syllabus. Inequalities are essentially used for intra-mathematical applications: in the resolution of problems modelled by equations or systems of equations, existence

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<sup>3</sup> As was shown by research on curricular changes in elementary analysis or calculus, analysing the effects of curricular changes requires a long term perspective : after a reform, during many years, what lives in classrooms is a subtle mixture of the old and new curriculum ; this creates an intermediary culture and long term effects, only visible when the ancient values tend to vanish, may be quite different from short term effects.

conditions for variables and parameters have to be taken into account and these lead to inequalities. This is thus their main ecological “niche”.

### *The “functional and structural stage”*

In the second stage, the context becomes structural and functional : structural because the object “inequality” is associated with the study of order structures (in 1971, middle school students are exposed to the proof that  $\mathbb{R}$  is a totally ordered field) and functional since this object is depending upon the notion of function, which is introduced very early. A new organisation of the syllabus replaces the old one. For instance, the grade 8 syllabus is divided into four parts: « I. Relations, decimal numbers and approach to real numbers, II. Geometry of the straight line, III. Plane geometry », and the grade 9 syllabus into three parts: « I. Real numbers, algebraic computation and numerical functions, II. Euclidean plane, III. Euclidean plane geometry ». Inequalities are introduced at grade 8 in relationship with the fact that  $\mathbb{R}$  is a totally ordered field, and, at grade 9, connection is made with the notion of interval. Equations and inequalities remain close objects but they live now together in a new “habitat”, that of numerical functions, which becomes a core notion for the curricular organisation<sup>4</sup>. This new “habitat” reinforces the role played by the graphical register in the solving of inequalities. It also situates the work on inequalities in a new ecological chain leading to their investment in elementary analysis at more advanced levels. Simultaneously, applications for inequalities change, with both intra and extra mathematical components. From an internal point of view, inequalities become a tool for preparing calculus work as pointed out above, from an external point of view, problems linked to human, social and economic sciences, enter the field, especially linear programming problems. But these problems turn out to be too complex and they soon disappear at this grade level.

### *The third stage*

In the third stage, as pointed out by Chevallard (Chevallard, 1985, 1989), algebra tends to be a « vanishing point » : its object is no longer the study of equations as in the classical period, neither the study of numerical structures as in the period of modern mathematics but it dilutes itself in different sections, mainly the section « Numerical Works » (inequalities are part of that section). Emblematic objects of the classic algebra, such as equations, are still present in the syllabus, and inequalities come after these, inequalities still appear linked to the relationship of order, but a new coherence for this area is still expecting to be built: algebra appears as an empirical domain deprived from structure. In the syllabus, emphasis is put on its modelling role.

In this context, inequalities appear as tools for modelling “daily-life” problems. Such choices a priori meet scientific needs (acknowledgement of the scientific importance of modelling activities in mathematical work and development), educational views (linking mathematics with concrete issues, with daily-life problems in order to evidence their usefulness and favour empirical and experimental approaches).

### *Further comments*

These three contexts, briefly described, shape the teaching of inequalities all over the century but a more detailed analysis is necessary in order to understand their precise effects. This analysis is undertaken by T. Assude (Assude, 2000) in terms of praxeologies, looking at the way tasks, techniques and technologies evolve along the century and how this evolution can be related to the contextual characteristics described above. This analysis shows interesting moves. For instance, thirteen different types of tasks are identified and seven types of techniques, but only one task: solving an first degree inequality with one unknown, is stable all

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<sup>4</sup> The evolution is not so abrupt as this schematic description presents it. The functional world progressively entered the scene from the 1947 reform

along the century. The number of tasks obeys a movement of extension-reduction: four in the 1902 reform, eight in 1960, seven in 1971, five in 1977 and 1985, 2 only in 1997. The drastic reduction we observe now is specially problematic. It is not an isolated point in mathematics teaching and reflects the modes of adaptation of the French educational system to the difficulties it meets. This adaptation leads to progressive reduction in contents and to a drastic reduction of the technical ambitions of mathematics teaching. Such choices are legitimated by arguments which can appear valuable: “teach less in order to teach better”, “favour conceptual understanding”, but evolution tends to show that teaching less does not necessarily promote better learning and that, in mathematical activity, the technical and conceptual dimensions cannot be separated so easily: reducing drastically the possibilities for technical work can limit the field of mathematical experiences one can live and limit understanding<sup>5</sup>. We will come back to this point in the discussion.

## Discussion and comments

In this text, we have presented three different research projects which approach learning and teaching issues in algebra from different perspectives but within a global coherent anthropological approach. What can we learn from this as regards the issues raised in the discussion document? We would like to briefly stress some points which, in our opinion, can offer a valuable contribution to the reflection.

The diversity of personal and institutional or cultural relationships with algebra. Reflecting about the future of teaching and learning algebra, even when one only considers elementary algebra requires a good awareness of this diversity and of the necessary multidimensionality of algebraic knowledge. This cannot be correctly approached through simplistic and hierarchic views, it is much more than that. In our opinion, B. Grugeon’s research evidences in a convincing way the fine grained analysis which is necessary if one wants to approach the personal relationship a student has with algebra, detect in her or his functioning germs for entering algebraic thinking and practices, efficiently help her or him overcome resistant difficulties by identifying underlying coherence, taking into account the specificity of her or his culture and needs.

The complexity of competencies required from teachers if we want these make their student benefit from the advances of didactic research in algebra. A. Lenfant’s thesis evidences this fact and, at the same time, shows how current epistemology of algebra in the culture but even in the educational system creates resistant obstacles to changes and professional development as it can be thought today. Elementary algebra tend to be restricted to a stereotyped and limited set of problems which do not reflect the epistemological values of this mathematical field. The role it has to play in the development of students’ mathematics rationality is negated in a culture where geometry is the temple for rationality. Algebraic techniques are mostly reduced to their syntactic part, transforming algebra in a world of legal rules. This vision prevents the development of the dialectic game between syntax and semantics which is characteristic of algebraic practices as soon as one goes beyond simple routine tasks. But, at the same time, A. Lenfant’s thesis shows that, if we make the effort of building adequate tools for analysing teachers’ functioning, we can identify different germs for the development of a professional expertise in that area, and subtle but promising evolution in teacher-students which are still in a transition phase. Once more, even if regularities appear, there is no uniformity. Germs are different from one teacher-student to another one, according to her (his) personal sensibility and biography.

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<sup>5</sup> For instance, limitation to only one type of inequality strongly reduces the field of situations accessible to mathematical modelling and reinforces the tendency to standardisation of modelling activities, which favour neither the understanding of modelling processes nor the understanding of algebraic concepts.

The third point, we would like to mention is evidenced by the T. Assude's research. Curricular constraints strongly shape the relationships with algebra which can live in a given institution at a given grade and the way these can evolve along time. Understanding these constraints imposes to go beyond their most apparent curricular features. On the one hand, there are syllabuses and the ambitions they express under more global curricular umbrellas, on the other hand, there are praxeologies which are potentially or effectively associated with the syllabus, in textbooks and in classrooms. The links between these two categories are far from being simple and the ideological discourse which goes with the presentation of the syllabus does not necessarily foster an adequate analysis. In France, today, for instance, there is a strong tendency to reduce technical ambitions as regards algebra, within the frame of a global discourse which opposes technical and conceptual activities in mathematics and wants to promote conceptual understanding. This is a too simplistic opposition: technical work, if not routine work, is also essential for understanding and conceptualising. Conceptual activities do not live in the air, they live supported by the manipulation of different categories of "ostensive", with the meaning given to this term in (Bosch & Chevallard, 1999), they rely on technical work. If ambitions at this level are drastically reduced, the effect can be contrary to the intention. There is a tendency to equate technical work and work without reflection and intelligence which is specially pernicious in algebra.

The last point we would like to mention deals with the relationships between research and practice. B. Grugeon's research clearly shows that transforming tools which have proved their efficiency in research into effective tools for teachers is not a trivial task. But this research also proves that transposition is possible and can be helped by technological advances. This is certainly costly and, in our case, was only possible thanks to the collaboration with computer scientists, but we have some reasons to be optimistic. The first experiments carried out this year seem to show that the tool which exists today (see the website: <http://pepite.lemans.fr>) is accessible to teachers and, beyond that, that it seems to be an effective tool in order to promote teachers' professional development in that area.

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